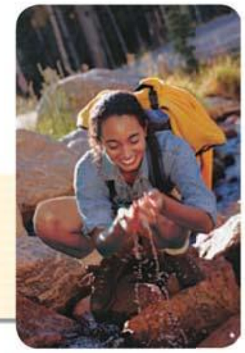
5.6 Inequalities in Two Triangles and Indirect Proof



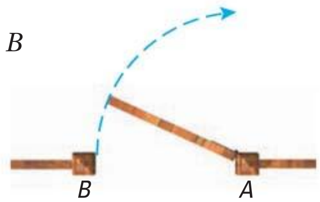
Before You used inequalities to make comparisons in one triangle.

Now You will use inequalities to make comparisons in two triangles.

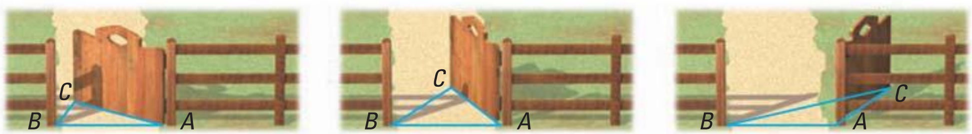
Why? So you can compare the distances hikers traveled, as in Ex. 22.

Key Vocabulary
- **indirect proof**
- **included angle,** *p. 240*

Imagine a gate between fence posts A and B that has hinges at A and swings open at B.



As the gate swings open, you can think of $\triangle ABC$, with side $\overline{AC}$ formed by the gate itself, side $\overline{AB}$ representing the distance between the fence posts, and side $\overline{BC}$ representing the opening between post B and the outer edge of the gate.

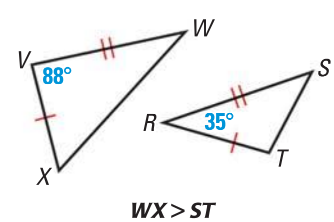


Notice that as the gate opens wider, both the measure of $\angle A$ and the distance CB increase. This suggests the *Hinge Theorem*.

THEOREMS — For Your Notebook

THEOREM 5.13 Hinge Theorem

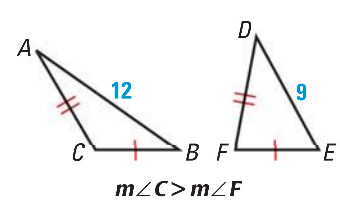
If two sides of one triangle are congruent to two sides of another triangle, and the included angle of the first is larger than the included angle of the second, then the third side of the first is longer than the third side of the second.



Proof: Ex. 28, p. 341

THEOREM 5.14 Converse of the Hinge Theorem

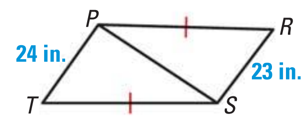
If two sides of one triangle are congruent to two sides of another triangle, and the third side of the first is longer than the third side of the second, then the included angle of the first is larger than the included angle of the second.



Proof: Example 4, p. 338

EXAMPLE 1 Use the Converse of the Hinge Theorem

Given that $\overline{ST} \cong \overline{PR}$, how does $\angle PST$ compare to $\angle SPR$?



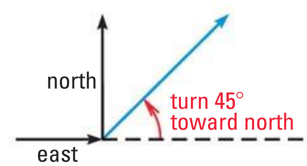
Solution

You are given that $\overline{ST} \cong \overline{PR}$ and you know that $\overline{PS} \cong \overline{PS}$ by the Reflexive Property. Because 24 inches > 23 inches, $PT > RS$. So, two sides of $\triangle STP$ are congruent to two sides of $\triangle PRS$ and the third side in $\triangle STP$ is longer.

▶ By the Converse of the Hinge Theorem, $m\angle PST > m\angle SPR$.

EXAMPLE 2 Solve a multi-step problem

BIKING Two groups of bikers leave the same camp heading in opposite directions. Each group goes 2 miles, then changes direction and goes 1.2 miles. Group A starts due east and then turns 45° toward north as shown. Group B starts due west and then turns 30° toward south.



Which group is farther from camp? Explain your reasoning.

Solution

Draw a diagram and mark the given measures. The distances biked and the distances back to camp form two triangles, with congruent 2 mile sides and congruent 1.2 mile sides. Add the third sides of the triangles to your diagram.

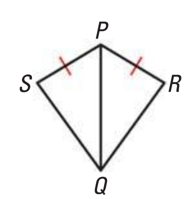
Next use linear pairs to find and mark the included angles of 150° and 135°.

▶ Because 150° > 135°, Group B is farther from camp by the Hinge Theorem.

AnimatedGeometry at classzone.com

✓ GUIDED PRACTICE for Examples 1 and 2

Use the diagram at the right.



1. If $PR = PS$ and $m\angle QPR > m\angle QPS$, which is longer, $\overline{SQ}$ or $\overline{RQ}$?
2. If $PR = PS$ and $RQ < SQ$, which is larger, $\angle RPQ$ or $\angle SPQ$?
3. **WHAT IF?** In Example 2, suppose Group C leaves camp and goes 2 miles due north. Then they turn 40° toward east and continue 1.2 miles. *Compare* the distances from camp for all three groups.

INDIRECT REASONING Suppose a student looks around the cafeteria, concludes that hamburgers are not being served, and explains as follows.

> *At first I assumed that we are having hamburgers because today is Tuesday and Tuesday is usually hamburger day.*
>
> *There is always ketchup on the table when we have hamburgers, so I looked for the ketchup, but I didn't see any.*
>
> *So, my assumption that we are having hamburgers must be false.*

The student used *indirect* reasoning. So far in this book, you have reasoned *directly* from given information to prove desired conclusions.

In an **indirect proof**, you start by making the temporary assumption that the desired conclusion is false. By then showing that this assumption leads to a logical impossibility, you prove the original statement true *by contradiction*.

KEY CONCEPT — *For Your Notebook*

How to Write an Indirect Proof

STEP 1 **Identify** the statement you want to prove. **Assume** temporarily that this statement is false by assuming that its opposite is true.

STEP 2 **Reason** logically until you reach a contradiction.

STEP 3 **Point out** that the desired conclusion must be true because the contradiction proves the temporary assumption false.

EXAMPLE 3 Write an indirect proof

Write an indirect proof that an odd number is not divisible by 4.

GIVEN ▶ x is an odd number.

PROVE ▶ x is not divisible by 4.

Solution

STEP 1 Assume temporarily that x is divisible by 4. This means that $\frac{x}{4} = n$ for some whole number n. So, multiplying both sides by 4 gives $x = 4n$.

STEP 2 If x is odd, then, by definition, x cannot be divided evenly by 2. However, $x = 4n$ so $\frac{x}{2} = \frac{4n}{2} = 2n$. We know that $2n$ is a whole number because n is a whole number, so x *can* be divided evenly by 2. This contradicts the given statement that x is odd.

STEP 3 Therefore, the assumption that x is divisible by 4 must be false, which proves that x is not divisible by 4.

READ VOCABULARY
You have reached a *contradiction* when you have two statements that cannot both be true at the same time.

✓ GUIDED PRACTICE for Example 3

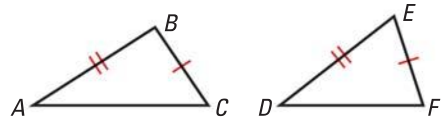
4. Suppose you wanted to prove the statement "If $x + y \neq 14$ and $y = 5$, then $x \neq 9$." What temporary assumption could you make to prove the conclusion indirectly? How does that assumption lead to a contradiction?

EXAMPLE 4 Prove the Converse of the Hinge Theorem

Write an indirect proof of Theorem 5.14.

GIVEN ▶ $\overline{AB} \cong \overline{DE}$
$\overline{BC} \cong \overline{EF}$
$AC > DF$

PROVE ▶ $m\angle B > m\angle E$



Proof Assume temporarily that $m\angle B \not> m\angle E$. Then, it follows that either $m\angle B = m\angle E$ or $m\angle B < m\angle E$.

Case 1 If $m\angle B = m\angle E$, then $\angle B \cong \angle E$. So, $\triangle ABC \cong \triangle DEF$ by the SAS Congruence Postulate and $AC = DF$.

Case 2 If $m\angle B < m\angle E$, then $AC < DF$ by the Hinge Theorem.

Both conclusions contradict the given statement that $AC > DF$. So, the temporary assumption that $m\angle B \not> m\angle E$ cannot be true. This proves that $m\angle B > m\angle E$.

✓ GUIDED PRACTICE for Example 4

5. Write a temporary assumption you could make to prove the Hinge Theorem indirectly. What two cases does that assumption lead to?

5.6 EXERCISES

HOMEWORK KEY
○ = **WORKED-OUT SOLUTIONS** on p. WS1 for Exs. 5, 7, and 23
★ = **STANDARDIZED TEST PRACTICE** Exs. 2, 9, 19, and 25

SKILL PRACTICE

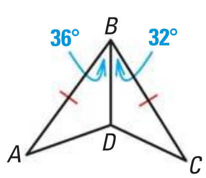
1. **VOCABULARY** Why is indirect proof also called *proof by contradiction*?

2. ★ **WRITING** *Explain* why the name "Hinge Theorem" is used for Theorem 5.13.

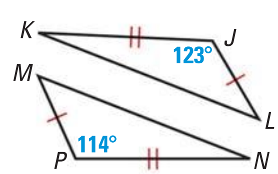
EXAMPLE 1 on p. 336 for Exs. 3–10

APPLYING THEOREMS **Copy and complete with <, >, or =. *Explain*.**

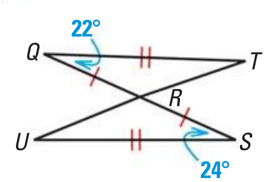
3. AD __?__ CD



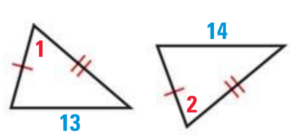
4. MN __?__ LK



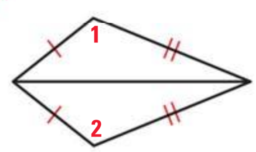
5. TR __?__ UR



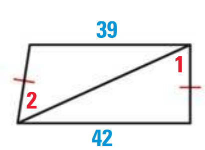
6. $m\angle 1$ __?__ $m\angle 2$



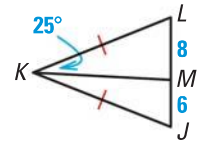
7. $m\angle 1$ __?__ $m\angle 2$



8. $m\angle 1$ __?__ $m\angle 2$



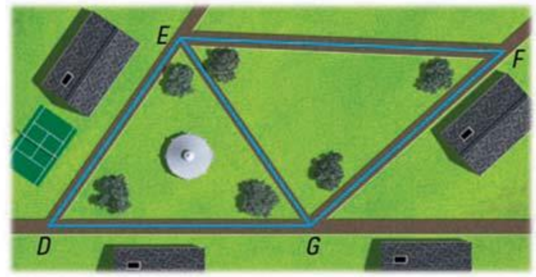
9. ★ **MULTIPLE CHOICE** Which is a possible measure for $\angle JKM$?



Ⓐ 20° Ⓑ 25°

Ⓒ 30° Ⓓ Cannot be determined

10. **USING A DIAGRAM** The path from E to F is longer than the path from E to D. The path from G to D is the same length as the path from G to F. What can you conclude about the angles of the paths? *Explain* your reasoning.



EXAMPLES 3 and 4 on p. 337–338 for Exs. 11–13

STARTING AN INDIRECT PROOF **In Exercises 11 and 12, write a temporary assumption you could make to prove the conclusion indirectly.**

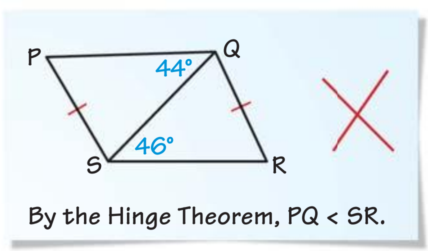
11. If x and y are odd integers, then xy is odd.

12. In $\triangle ABC$, if $m\angle A = 100°$, then $\angle B$ is not a right angle.

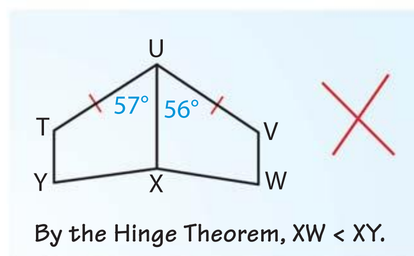
13. **REASONING** Your study partner is planning to write an indirect proof to show that $\angle A$ is an obtuse angle. She states "Assume temporarily that $\angle A$ is an acute angle." What has your study partner overlooked?

ERROR ANALYSIS ***Explain*** **why the student's reasoning is not correct.**

14.

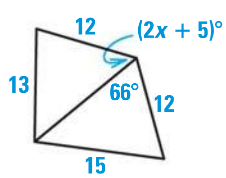


15.

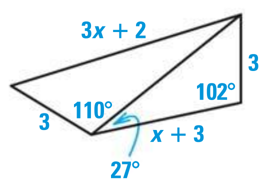


ALGEBRA **Use the Hinge Theorem or its converse and properties of triangles to write and solve an inequality to describe a restriction on the value of x.**

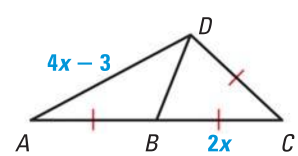
16.



17.



18.



19. ★ **SHORT RESPONSE** If $\overline{NR}$ is a median of $\triangle NPQ$ and $NQ > NP$, *explain* why $\angle NRQ$ is obtuse.

20. **ANGLE BISECTORS** In $\triangle EFG$, the bisector of $\angle F$ intersects the bisector of $\angle G$ at point H. *Explain* why $\overline{FG}$ must be longer than $\overline{FH}$ or $\overline{HG}$.

21. **CHALLENGE** In $\triangle ABC$, the altitudes from B and C meet at D. What is true about $\triangle ABC$ if $m\angle BAC > m\angle BDC$? *Justify* your answer.

PROBLEM SOLVING

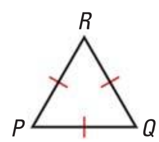
EXAMPLE 2
on p. 336
for Ex. 22

22. HIKING Two hikers start at the visitor center. The first hikes 4 miles due west, then turns 40° toward south and hikes 1.8 miles. The second hikes 4 miles due east, then turns 52° toward north and and hikes 1.8 miles. Which hiker is farther from camp? *Explain* how you know.

@HomeTutor for problem solving help at classzone.com

EXAMPLES
3 and 4
on pp. 337–338
for Exs. 23–24

23. INDIRECT PROOF Arrange statements A–E in order to write an indirect proof of the corollary: If $\triangle ABC$ is *equilateral*, then it is *equiangular*.



GIVEN ▶ $\triangle PQR$ is equilateral.

A. That means that for some pair of vertices, say P and Q, $m\angle P > m\angle Q$.

B. But this contradicts the given statement that $\triangle PQR$ is equilateral.

C. The contradiction shows that the temporary assumption that $\triangle PQR$ is not equiangular is false. This proves that $\triangle PQR$ is equiangular.

D. Then, by Theorem 5.11, you can conclude that $QR > PR$.

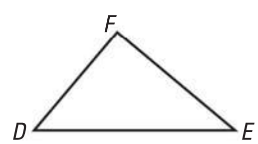
E. Temporarily assume that $\triangle PQR$ is not equiangular.

@HomeTutor for problem solving help at classzone.com

24. PROVING THEOREM 5.11 Write an indirect proof of Theorem 5.11, page 328.

GIVEN ▶ $m\angle D > m\angle E$

PROVE ▶ $EF > DF$



Plan for Proof In Case 1, assume that $EF < DF$. In Case 2, assume that $EF = DF$.

25. ★ EXTENDED RESPONSE A scissors lift can be used to adjust the height of a platform.

a. Interpret As the mechanism expands, $\overline{KL}$ gets longer. As KL increases, what happens to $m\angle LNK$? to $m\angle KNM$?

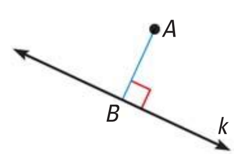
b. Apply Name a distance that decreases as $\overline{KL}$ gets longer.

c. Writing *Explain* how the adjustable mechanism illustrates the Hinge Theorem.

26. PROOF Write a proof that the shortest distance from a point to a line is the length of the perpendicular segment from the point to the line.

GIVEN ▶ Line k; point A not on k; point B on k such that $\overline{AB} \perp k$

PROVE ▶ $\overline{AB}$ is the shortest segment from A to k.



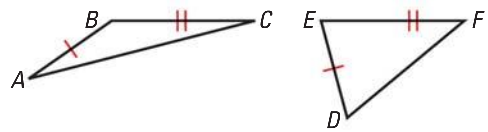
Plan for Proof Assume that there is a shorter segment from A to k and use Theorem 5.10 to show that this leads to a contradiction.

27. **USING A CONTRAPOSITIVE** Because the contrapositive of a conditional is equivalent to the original statement, you can prove the statement by proving its contrapositive. Look back at the conditional in Example 3 on page 337. Write a proof of the contrapositive that uses direct reasoning. How is your proof similar to the indirect proof of the original statement?

28. **CHALLENGE** Write a proof of Theorem 5.13, the Hinge Theorem.

GIVEN ▶ $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, $m\angle ABC > m\angle DEF$

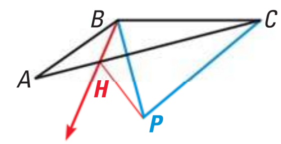
PROVE ▶ $AC > DF$



Plan for Proof

1. Because $m\angle ABC > m\angle DEF$, you can locate a point P in the interior of $\angle ABC$ so that $\angle CBP \cong \angle FED$ and $\overline{BP} \cong \overline{ED}$. Draw $\overline{BP}$ and show that $\triangle PBC \cong \triangle DEF$.
2. Locate a point H on $\overline{AC}$ so that $\overrightarrow{BH}$ bisects $\angle PBA$ and show that $\triangle ABH \cong \triangle PBH$.
3. Give reasons for each statement below to show that $AC > DF$.

$AC = AH + HC = PH + HC > PC = DF$



MIXED REVIEW

PREVIEW
Prepare for Lesson 6.1 in Exs. 29–31.

Write the conversion factor you would multiply by to change units as specified. *(p. 886)*

29. inches to feet
30. liters to kiloliters
31. pounds to ounces

Solve the equation. Write a reason for each step. *(p. 105)*

32. $1.5(x + 4) = 5(2.4)$
33. $-3(-2x + 5) = 12$
34. $2(5x) = 3(4x + 6)$

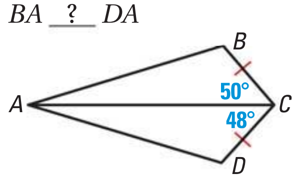
35. Simplify the expression $\dfrac{-6xy^2}{21x^2y}$ if possible. *(p. 139)*

QUIZ *for Lessons 5.5–5.6*

1. Is it possible to construct a triangle with side lengths 5, 6, and 12? If not, *explain* why not. *(p. 328)*
2. The lengths of two sides of a triangle are 15 yards and 27 yards. *Describe* the possible lengths of the third side of the triangle. *(p. 328)*
3. In $\triangle PQR$, $m\angle P = 48°$ and $m\angle Q = 79°$. List the sides of $\triangle PQR$ in order from shortest to longest. *(p. 328)*

Copy and complete with <, >, or =. *(p. 335)*

4. BA __?__ DA



5. $m\angle 1$ __?__ $m\angle 2$